On a spherical code in the space of spherical harmonics

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Abstract

In this short note we propose a new method for construction new nice arrangement on the sphere S^d using the spaces of spherical harmonic.

Keywords: E_8 lattice, spherical harmonics, spherical antipodal code, spherical designs.

1 Introduction

This paper is inspired by classical book J. H. Conway and N. J. A. Sloane [3] and recent paper of H. Cohn and A. Kumar [2]. The exceptional arrangement of points on the spheres are discussed there. Especially interesting are constructions coming from well known E_8 lattice and Leech lattice Λ_{24}. The main idea of the paper is to use these arrangements for construction new good arrangements in the spaces of spherical harmonics $\mathcal{H}_k^d$. Recently we have use dramatically the calculations in these spaces to obtain new asymptotic existence bounds for spherical designs, see [1]. Below we need a few facts on spherical harmonics. Let Δ be the Laplace operator in $\mathbb{R}^{d+1}$

$$\Delta = \sum_{j=1}^{d+1} \frac{\partial^2}{\partial x_j^2}.$$

We say that a polynomial P in $\mathbb{R}^{d+1}$ is harmonic if $\Delta P = 0$. For integer $k \geq 1$, the restriction to S^d of a homogeneous harmonic polynomial of degree

k is called a spherical harmonic of degree k. The vector space of all spherical harmonics of degree k will be denoted by $\mathcal{H}_k^d$ (see [4] for details). The dimension of $\mathcal{H}_k^d$ is given by

$$\dim \mathcal{H}_k^d = \frac{2k+d-1}{k+d-1}\binom{d+k-1}{k}.$$

Consider usual inner product in $\mathcal{H}_k^d$

$$\langle P, Q\rangle := \int_{S^d} P(x)Q(x)d\mu_d(x),$$

where $\mu_d(x)$ is normalized Lebesgue measure on the unit sphere S^d. Now, for each point $x \in S^d$ there exists a unique polynomial $P_x \in \mathcal{H}_k^d$ such that

$$\langle P_x, Q\rangle = Q(x) \text{ for all } Q \in \mathcal{H}_k^d.$$

It is well known that $P_x(y) = g((x,y))$, where g is a corresponding Gegenbauer polynomial. Let G_x be normalized polynomial P_x, that is $G_x = P_x/g(1)^{1/2}$. Note that $\langle G_{x_1}, G_{x_2}\rangle = g((x_1,x_2))/g(1)$. So, if we have some arrangement $X = \{x_1,\dots,x_N\}$ on S^d with known distribution of inner products (x_i,x_j), then for each k we have corresponding set $G_X = \{G_{x_1},\dots,G_{x_N}\}$ in $\mathcal{H}_k^d$, also with known distribution of inner products. Using this construction we will obtain in the next section the optimal antipodal spherical $(35, 240, 1/7)$ code from minimal vectors of E_8 lattice. Here is a definition.

Definition 1. The antipodal set $X = \{x_1,\dots,x_N\}$ on S^d is called antipodal spherical $(d+1, N, a)$ code, if $|(x_i,x_j)| \le a$, for some $a > 0$ and for all $x_i, x_j \in X$, $i \ne j$, which are not antipodal. Such code is called optimal if for any antipodal set $Y = \{y_1,\dots,y_N\}$ on S^d there exists $y_i, y_j \in Y$, $i \ne j$, which are not antipodal and $|(y_i,y_j)| \ge a$.
In the other words, antipodal spherical $(d+1, N, a)$ code is optimal if a is a minimal possible number for fixed N, d.

2 Construction and the proof of optimality

Let $X = \{x_1,\dots,x_{120}\}$ be any subset of 240 normalized minimal vectors of E_8 lattice, such that no pair of antipodal vectors presents in X. Take in the space $\mathcal{H}_2^7$ the polynomials

$$G_{x_i}(y) = g_2((x_i,y)), \qquad i = \overline{1,\dots,120},$$

where $g_2(t) = \frac{8}{7}t^2 - \frac{1}{7}$ is a corresponding normalized Gegenbauer polynomial. Since $(x_i, x_j) = 0$ or $\pm 1/2$, for $i \neq j$, then $\langle G_{x_i}, G_{x_j} \rangle = g_2((x_i, x_j)) = \pm 1/7$! It looks really like a mystery the fact that $|g_2((x_i, x_j))| = const$, for any different $x_i, x_j \in X$. But exactly this is essential for the proof of optimality of our code. Since, $dim\mathcal{H}_2^7 = 35$, then the points $G_{x_1}, \ldots, G_{x_{120}}, -G_{x_1}, \ldots, -G_{x_{120}}$ provide antipodal spherical $(35, 240, 1/7)$ code. Here is a proof of optimality. Take arbitrary antipodal set of points $Y = \{y_1, \ldots, y_{240}\}$ in $\mathbb{R}^{35}$. Then, the inequality

$$\frac{1}{240^2} \sum_{i,j=1}^{240} (y_i, y_j)^2 \geq 1/35,$$

implies that $(y_i, y_j)^2 \geq 1/49$, for some $y_i, y_j \in Y$, $i \neq j$, which are not antipodal. This immediately gives us an optimality of our construction. The other reason why it works, that is our set is also spherical 3-design in $\mathbb{R}^{35}$. We are still not able generalize this construction even for Leech lattice Λ_{24}. We also don't know whether the construction described above is an optimal spherical $(35, 240, 1/7)$ code.

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References

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